5. Multi-Dimensional Conduction

In this lecture, we will deal with heat conduction problem significant in more than one-dimension. This approach should be used as the applications imply. First, several alternatives are developed to deal with two-dimensional, steady state conduction. Then as we reach the numerical approach, we can extend its use for a three-dimensional problem.

5.1 Two-Dimensional and Steady-State Conduction

Under the assumptions of two dimensional steady state conduction the general heat equation is reduced to

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = 0 \qquad (5.1)$$

Now we have two goals for solving the above equation, the first is to determine the temperature distribution across the flied which became a function in the two coordinates x and y T(x, y), then to determine the heat fluxes q_x'' and q_y'' in the two direction x and y respectively.

There are many techniques for solving the Equation 5.1 including: analytical, graphical and numerical solution (finite element and finite difference approaches).

The analytical solution is much more difficult than that of the one dimensional steady state conduction since the equations are partial differential equations, the mathematical solution is very difficult and is limited to a set of simple geometries, on the other hand the exact solution gives the dependent variable T as a continuous function in the independents (x, y) and the solution could be determined at any point of interest in the field of study.

On the other hand the graphical and the numerical solution gives an approximate solution at discrete points in the medium, as the graphical and numerical can solve complex geometries, they are more widely used for the multidimensional conduction problems.

5.1.1 The Method for Separation of Variables

Solving Equation 5.1 for a rectangular plate as shown in Figure 5.1, with three boundaries maintained at T_1, while the fourth side is maintained at T_2, where $T_2 \neq T_1$, the solution of this problem should give the temperature distribution T(x, y) at any point in the solution domain. For solution purpose, the following transformation is done.

$$\theta = \frac{T - T_1}{T_2 - T_1} \qquad (5.2)$$

And thus the heat equation yields

$$\frac{\partial^2\theta}{\partial x^2}+\frac{\partial^2\theta}{\partial y^2}=0 \qquad (5.3)$$

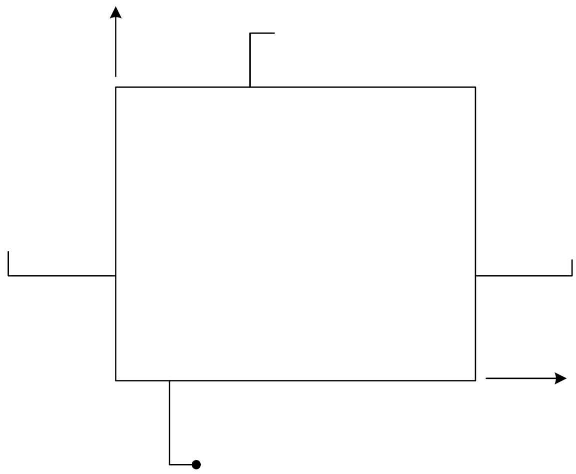


Figure5.1 Geometric configuration for the method of separation of variables

Since the equation is second order in both x and y, two boundary equations are required for each coordinate which are

$$\begin{aligned}\theta(0,y)&=0 \quad\text{and}\quad \theta(x,0)=0\\ \theta(L,y)&=0 \quad\text{and}\quad \theta(x,W)=1\end{aligned} \qquad (5.4)$$

The separation of variables technique is applied by assuming that the required function is the product of the two functions X (x) and Y (y)

$$\theta(x,y)=X(x)\cdot Y(y) \qquad (5.5)$$

Substituting in Equation 5.3 and dividing by XY

$$-\frac{1}{X}\frac{d^2X}{dx^2}=\frac{1}{Y}\frac{d^2Y}{dy^2} \qquad (5.6)$$

It is evident that Equation 5.6 is separable as the left-hand-side depends only on x, and the right-hand-side depends only on y. Therefore, the equality can only apply if both sides are equal to the same constant, λ^2, called the separation constant. Using this constant, Equation 5.6 can yield the following equations

$$\frac{d^2X}{dx^2}+\lambda^2X=0 \qquad (5.7)$$

$$\frac{d^2Y}{dy^2}-\lambda^2Y=0 \qquad (5.8)$$

Then the partial differential equation is converted to two second order ordinary differential equations. The value of λ^2 must not be negative nor zero in order that the solution satisfies the prescribed boundary equation.

The solutions equation of the above ODE gives

$$X = C_1 \cos \lambda x + C_2 \sin \lambda x \qquad (5.9)$$

$$Y = C_3 e^{-\lambda y} + C_4 e^{+\lambda y} \qquad (5.10)$$

The general solution of the heat equation is

$$\theta = \left(C_1 \cos \lambda x + C_2 \sin \lambda x\right)\left(C_3 e^{-\lambda y} + C_4 e^{+\lambda y}\right) \qquad (5.11)$$

Then applying the boundary condition that θ (0, y) = 0, we get that $C_1 = 0$,

Then the condition that θ (x, 0) = 0, we get

$$C_2 \sin \lambda x\,(C_3 + C_4) = 0 \qquad (5.12)$$

The above equation is satisfied by either $C_3 = -C_4$ or $C_2 = 0$, but if we consider the solution that $C_2 = 0$ this will eliminate the solution dependency on x coordinate, which is a refused solution, thus the first solution is chosen $C_3 = -C_4$, applying the condition that θ (L, y) = 0, we get:

$$C_2 C_4 \sin \lambda L\,(e^{\lambda y} - e^{-\lambda y}) = 0 \qquad (5.13)$$

The only acceptable solution is that sin (λ L) = 0, this is satisfied for the values of

$$\lambda = \frac{n\pi}{L} \qquad \text{where } n = 1, 2, 3, \ldots$$

$$\therefore \theta = C_2 C_4 \sin \frac{n\pi x}{L}\left(e^{n\pi y/L} - e^{-n\pi y/L}\right) \qquad (5.14)$$

Rearranging

$$\theta(x, y) = C_n \sin \frac{n\pi x}{L} \sinh \frac{n\pi y}{L} \qquad (5.15)$$

Where C_n is a combined constant Equation 5.16 has an infinite number of solutions depending on n, however it is a linear problem. Thus a more general solution may be obtained by superposing all the solutions as

$$\theta(x, y) = \sum_{n=1}^{\infty} C_n \sin \frac{n\pi x}{L} \sinh \frac{n\pi y}{L} \qquad (5.16)$$

Now in order to determine C_n the remaining boundary condition should be applied

$$\theta(x, W) = 1 = \sum_{n=1}^{\infty} C_n \sin \frac{n\pi x}{L} \sinh \frac{n\pi y}{L} \qquad (5.17)$$

An analogous infinite series expansion is used in order to evaluate the value of C_n resulting that

$$C_n = \frac{2[(-1)^{n+1} + 1]}{n\pi \sinh(n\pi W / L)}$$

Substituting in Equation 5.16 we get

$$\theta(x, y) = \frac{2}{\pi}\sum_{n=1}^{\infty}\frac{[(-1)^{n+1}+1]}{n}\sin\frac{n\pi x}{L}\,\frac{\sinh(n\pi y / L)}{\sinh(n\pi W / L)} \qquad (5.18)$$

Then we obtain the solution of the rectangular shape in terms of θ as represented in the Figure 5.2 in the form of Isotherms for the schematic of the rectangular plate.

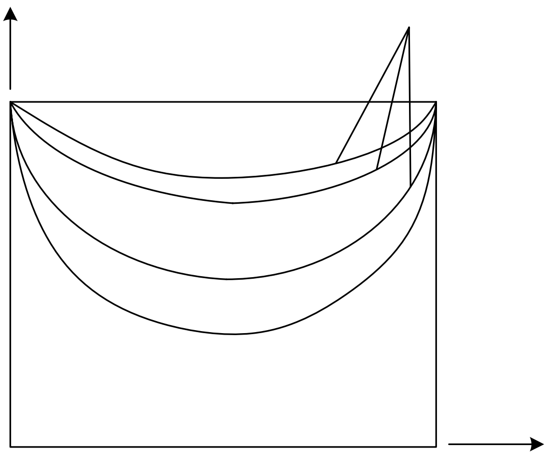


Figure 5.2 Isotherms for two-dimensional conduction in a rectangular plate

5.1.2 The Graphical Method

The graphical approach is applied for two dimensional conduction problems with adiabatic and isothermal boundaries, it has been used as a first estimate for the temperature distribution and to develop a guess for the physical nature of the temperature field and heat flux in the medium.

The idea of the graphical solution comes from the fact that the constant temperature lines must be perpendicular on the direction of heat flow, the objective in this method is to form a symmetrical network of isotherms and heat flow lines (adiabatic) which is called plot flux.

Consider a square, two-dimensional channel whose inner and outer surfaces are maintained at T_1 and T_2 respectively as shown in Figure 5.3(a). The plot flux and isothermal lines are shown in Figure 5.3(b).

The procedure for constructing the flux plot is enumerated as follows:

1. The first step in any flux plot should be to identify all relevant lines of symmetry. Such lines are determined by thermal, as well as geometrical, conditions. For the square channel of Figure 5.3(a), such lines include the designated vertical, horizontal, and diagonal lines. For this system it is therefore possible to consider only one-eighth of the configuration, as shown in Figure 5.3(b).

2. Lines of symmetry are adiabatic in the sense that there can be no heat transfer in a direction perpendicular to the lines. They are therefore heat flow lines and should be treated as such. Since there is no heat flow in a direction perpendicular to a heat flow line, such a line can be termed adiabatic.

3. After all known lines of constant temperature associated with the system boundaries have been identified, an attempt should be made to sketch lines of constant temperature within the system. Note that isotherms should always be perpendicular to adiabatic lines.

4. The heat flow lines should then be drawn with an eye toward creating a network of curvilinear squares. This is done by having the heat flow lines and isotherms intersect at right angles and by requiring that all sides of each square be of approximately the same length. It is often impossible satisfy this second requirement exactly, and it is more realistic to strive for equivalence between the sums of the opposite sides of each square. Assigning the x coordinate to the direction of the flow and the y coordinate to the direction normal to this flow, the requirement may be expressed as:

$$\Delta x \equiv \frac{ab+cd}{2} \approx \Delta y \equiv \frac{ac+bd}{2} \qquad (5.19)$$

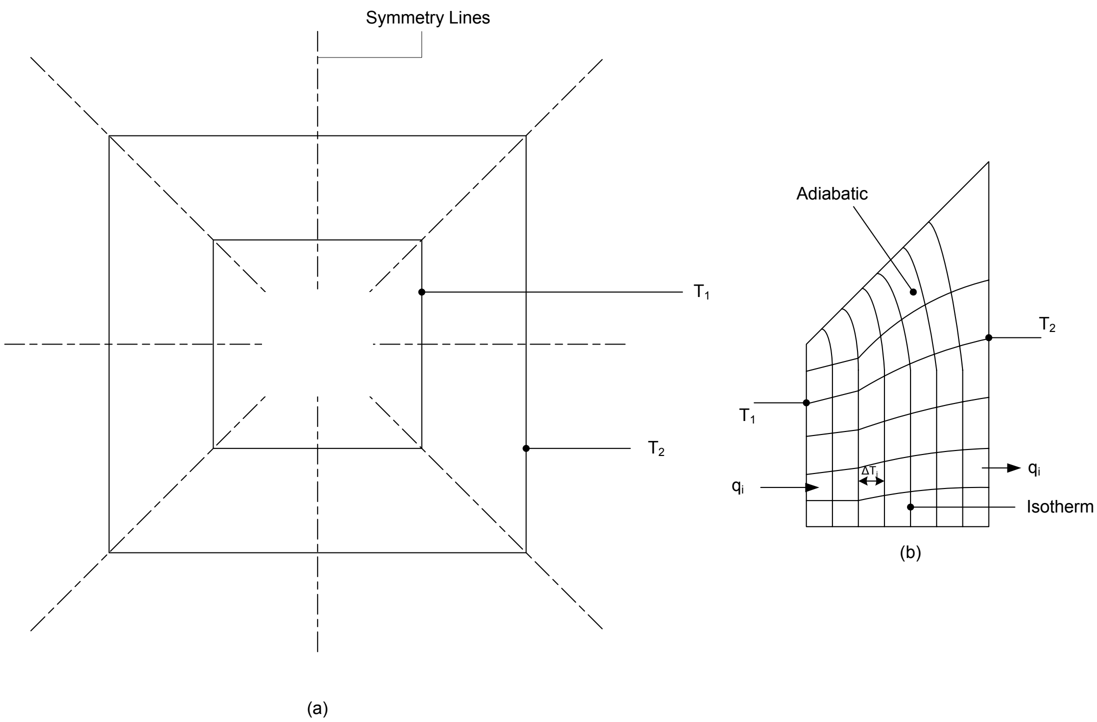


Figure 5.3 Graphical solution for Two-dimensional conduction in a square channel

The rate at which energy is conducted through a heat flow path, which is the region between adjoining adiabatic lines, is designated as q_i. If the flux plot is properly constructed, the value of q_i will be the same for all heat flow paths and the total heat transfer rate may be expressed as:

$$q = \sum_{i=1}^{M} q_i \qquad (5.20)$$

Where M is the number of heat flow paths associated with the plot. q_i may be expressed as:

$$q_i \approx kA_i \frac{\Delta T_j}{\Delta x} \approx k\left(\Delta y \cdot l\right)\frac{\Delta T_j}{\Delta x} \qquad (5.21)$$

Where ΔT_j is the temperature difference between successive isotherms, A, is the conduction heat transfer area for the heat flow path, and l is the length of the channel normal to the page. However, if the flux plot is properly constructed, the temperature increment is the same for all adjoining isotherms, and the overall temperature difference between boundaries, T_{1-2} may be expressed as

$$\Delta T_{1-2} = \sum_{j=1}^{N} \Delta T_j = N \Delta T_j \tag{5.22}$$

Where N is the total number of temperature increments. Combining Equations 5.20 to 5.22 and recognizing that Δx ≈ Δy for curvilinear squares, we obtain

$$q \approx \frac{Ml}{n} k \Delta T_{1-2} \tag{5.23}$$

The manner in which a flux plot may be used to obtain the heat transfer rate for a two-dimensional system is evident from Equation 5.23. The ratio of the number of heat flow paths to the number of temperature increments (the value of M/N) may be obtained from the plot. Recall that specification of N is based on step 3 of the foregoing procedure, and the value, which is an integer, may be made large or small depending on the desired accuracy. The value of M is then a consequence of following step 4. Note that M is not necessarily an integer, since a fractional lane may be needed to arrive at a satisfactory network of curvilinear squares. For the network of Figure 5.3(b), N = 7 and M = 6. Of course, as the network, or mesh, of curvilinear squares is made finer. N and 1W increase and the estimate of M/N becomes more accurate.

The above procedure is very time consuming, and can be done only for simple geometries. For this reasons, a simplification has been made by tabulating the shape factors for various two-dimensional problems in order to enable easier analysis of 2-D conductions.

Equation 5.23 may be used to define the shape factor, S, of a two-dimensional system as being the ratio (Ml / N). Hence, the heat transfer rate may be expressed as

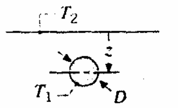
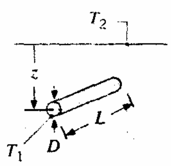
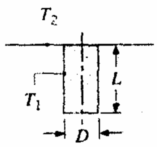
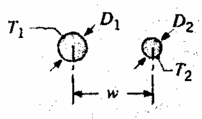
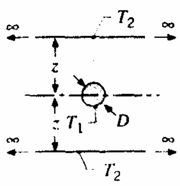
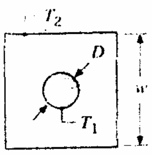
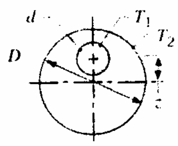
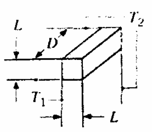
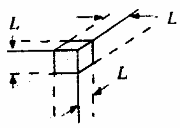
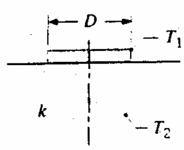
$$q = Sk\Delta T_{1-2} \tag{5.24}$$

From Equation 5.24, it also follows that a two-dimensional conduction resistance may he expressed as

$$R_{t,2-D\ cond} = \frac{1}{Sk} \tag{5.25}$$

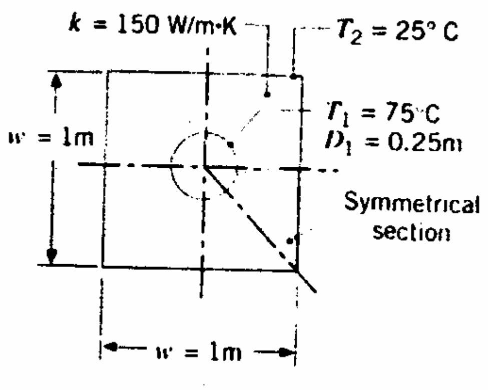
Shape factors for numerous two-dimensional systems and results are summarized in Table 5.1 for some common configurations. For each case, two-dimensional conduction is supposed to occur between boundaries that are maintained at uniform temperatures.

Table 5.1 Conduction shape factors for selected two-dimensional systems, $q=S\,k\,(T_1-T_2)$

System	Schematic	Restrictions	Shape Factor
Case 1. Isothermal sphere buried in a semi-infinite medium		z > D/2	$\frac{2\pi D}{1 - D/4z}$
Case 2. Horizontal isothermal cylinder of length L buried in a semi-infinite medium		L>>D L>>D z > 3D/2	$\frac{2\pi L}{\cosh^{-1}(2z/D)}$ $\frac{2\pi L}{\ln(4z/D)}$
Case 3. Vertical cylinder in a semi-infinite medium		L>>D	$\frac{2\pi L}{\ln(4L/D)}$
Case 4. Conduction between two cylinders of length L in infinite medium		L >> D_1,D_2 L >> w	$\frac{2\pi L}{\cosh^{-1}\left(\frac{4w^2 - D_1^2 - D_2^2}{2D_1D_2}\right)}$
Case 5. Horizontal circular cylinder of length L midway between parallel planes of equal length and infinite width		z >> D/2 L >> z	$\frac{2\pi L}{\ln(8z/\pi D)}$
Case 6. Circular cylinder of length L centred in a square solid of equal length			$\frac{2\pi L}{\ln(1.08\,w/D)}$
Case 7. Eccentric circular cylinder of length L in a cylinder of equal length		D > d L >> D	$\frac{2\pi L}{\cosh^{-1}\left(\frac{D^2 + d^2 - 4z^2}{2Dd}\right)}$
Case 8. Conduction through the edge of adjoining walls		D > L/5	0.54 D
Case 9. Conduction through corner of three walls with a temperature difference ΔT_{1-2} across the walls		L << length and width of wall	0.15 L
Case 10. Disk of diameter D and T_1 on a semi-infinite medium of thermal conductivity k and T_2		None	2 D

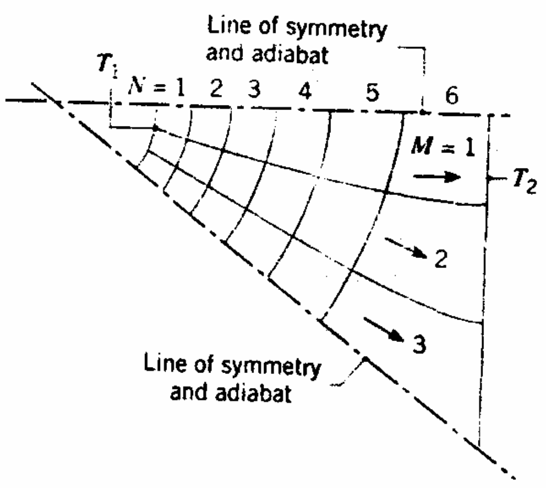
Example 5.1:
A hole of diameter D = 0.25 m is drilled through the centre of a solid block of square cross section with w = 1 m on a side. The hole is drilled along the length l = 2 m of the block, which has a thermal conductivity of k = 150 W/m K. A warm fluid passing through the hole maintains an inner surface temperature of T_1 75 °C, while the outer surface of the block is kept at T_2 = 25 °C.
1. Using the flux plot method, determine the shape factor for the system.
2. What is the rate of heat transfer through the block?

Solution:



Assumptions:
1. Steady-state Conditions.
2. Two-dimensional conduction.
3. Constant properties.
4. Ends of block are well insulated.

Analysis:
1. The flux plot may be simplified by identifying lines of symmetry and reducing the system to the one-eighth section shown in the schematic. Using a fairly coarse grid involving N = 6 temperature increments, the flux plot was generated. The resulting network of curvilinear squares is as follows.



With the number of heat flow lanes for the section corresponding to
M = 3, it follows that the shape factor for the entire block is
S = 8× (M × l / N) = 8 × (3 × 2 / 6) = 8 m

Where the factor of 8 results from the number of symmetrical sections.

The accuracy of this result may be determined by referring to Table 5.1, where, for the prescribed system, it follows that

$$S = \frac{2\pi L}{\ln(1.08 w/D)} = \frac{2\pi \times 2}{\ln(1.08 \times 1/0.25)} = 8.59 \text{ m}$$

Hence the result of the flux plot under predicts the shape factor by approximately 7%. Note that, although the requirement l >> w is not satisfied.

2. Using S = 8.59 m with Equation 5.24, the heat rate is

$q = S\,k\,(T_1 - T_2)$

$q = 8.59$ m × 150 W/m K (75 – 25) °C = 64.4 kW.

5.1.3 The Numerical Method

An alternative to the analytical and the graphical method is the numerical method, the numerical method involve different techniques such as finite difference, finite element and boundary –element method.

As stated in section 5.1.1 that the analytical solution gives the dependent variable T as a continuous function in the independent variables x and y. in contrast to the analytical solution the numerical solution converts the field or the system to discrete points at which the temperature is obtained. The domain is divided to small regions, referring to each region with a point at its centre as a reference point this point is termed nodal point or node, the network formed from these points is called nodal network, grid or mesh. Figure 5.4 presents a discritised domain along with the proper nomenclature.

The space or the difference between nodes is Δx in x direction and Δy in y direction, These nodes are numbered in both x and y direction and vary from 1 to m and n respectively and then conservation equation is applied to each point.

Using Taylor expansion the first and second derivatives are approximated to algebraic equations then we get number of simultaneous equations equal to number of nodes then this system of linear equations is solved either by a direct or indirect method to obtain the temperature value at each point.

Here are the algebraic equations expressing the derivatives in the conservation equation that is applied to each point.

$$\frac{\partial^2 T}{\partial x^2} = \frac{T_{m-1,n} - 2T_{m,n} + T_{m+1,n}}{(\Delta x)^2} \quad (5.27)$$

Similarly the derivative in the y direction is expressed as

$$\frac{\partial^2 T}{\partial y^2} = \frac{T_{m,n+1} - 2T_{m,n} + T_{m,n-1}}{(\Delta y)^2} \quad (5.28)$$

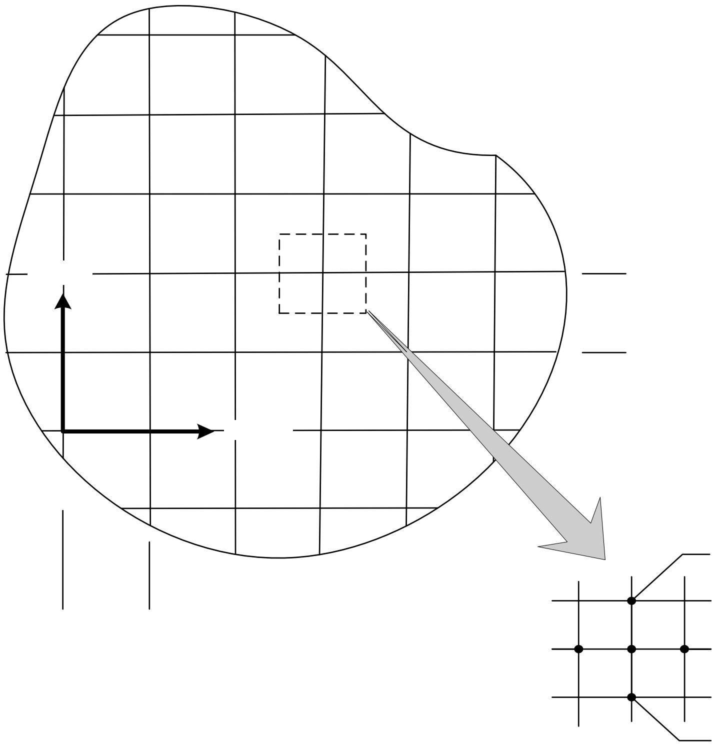


Figure 5.4 Discritised domains for a two-dimensional conduction problem

Substituting from Equations 5.27 and 5.28 into Equation 5.1 gives the discretised heat equation as

$$T_{m,n+1} + T_{m,n-1} + T_{m+1,n} + T_{m-1,n} - 4T_{m,n} = 0 \tag{5.29}$$

Now Equation 5.29 can be applied at each grid point so that a set of n × m simultaneous equations is formed. This set of equations can be solved either directly by matrix inversion method or indirectly by iterative procedures. Special care must be taken for boundary nodes, for this energy conservation should be applied.

5.2 Three-Dimensional and Steady-State Conduction

The problem of three-dimensional, steady-state conduction is a very tricky one. The most widely approach for such problems is the numerical approach. For this a discretised equation is developed and similar solution procedure as described in section 5.1.3 is performed to get temperature values at nodal points inside the domain.